International Journal of Theoretical Physics, Vol. 42, No. 2, February 2003 (© 2003)

The Qubits of Qunivac

David Ritz Finkelstein[1]

Received January 23, 2003

We distinguish between ontic and praxic formulations of quantum theory and adopt a praxic one. We formulate a reversible higher-order quantum logic in a large Clifford algebra Cliff(ι). We use it to describe the operation of the Quantum Universe As Computer (Qunivac). The qubits of Qunivac are associated with Clifford units with a real Clifford–Wilczek statistics. We encode the spin-$\frac{1}{2}$ Dirac equation on Qunivac in an exactly Lorentz-invariant ultraquantum space–time. Qunivac violates the canonical Heisenberg indeterminacy principle and locality in a way that should show up at high energies only. Qunivac accommodates a field theory.

KEY WORDS: Qunivac; Qubit; reversible logic; Clifford algebra; ultraquantum.

1. QUANTUM THEORY

There are today two inconsistent versions of quantum physics at large, which I call ontic and praxic. The praxic one is practiced widely, works well, and is professed by a small minority. The ontic is widely professed and almost never practiced. It is not self-consistent.

The discord appeared in the early days of quantum theory. Heisenberg invented a matrix mechanics, dealing solely with processes or operations, represented by matrices. Schrödinger invented a form of wave mechanics, an ontic theory in that it claimed to deal with real physical waves. The wave theory, however, was after-fitted with ad hoc translation rules that made it consistent with experiment and matrix mechanics. The resulting ontic theory propagated more rapidly than the praxic one—perhaps because it is more visualizable—and now has almost driven the praxic one out of the classroom.

Natural language not only conflicts with special relativity in its tense structure, but also conflicts with quantum physics in its predication structure. Ontism is built into natural language. In both cases natural language assumes a nonexistent now, an “is” that actually isn’t, a reality (or thingness) underlying actuality (or actness). Bacon would call this “is” an idol of the tribe (Bacon, 1994).

[1] School of Physics, Georgia Institute of Technology, Atlanta, Georgia; e-mail: df4@prism.gatech.edu

For illustration we juxtapose and align selected formulations of the two classes of theories that we call praxic and ontic, respectively (Finkelstein, 1996), taken from matrix mechanics and wave mechanics.

1.1. Praxic Theory

Ideal input and output processes are represented by vectors $|i\rangle$ *and dual vectors* $\langle o|$. *The transition probability from* $|i\rangle$ *to* $\langle o|$ *is*

$$P = \cos^2\theta := |\langle o|i\rangle|^2. \quad (1)$$

[The purist will object to vectors in a purely matrix mechanics. I use them to paraphrase the ontic theory most closely. The pure matrix form of the Malus–Born law gives the probability for a random quantum from a white source to go through a throughput process represented by a matrix T and be counted by an ideal black sink: $P = \mathrm{Tr}\, T^*T/\mathrm{Tr}\, 1$. Now we represent Malus' initial and final polarizing filters by projection matrices P_i, P_o in $T = P_o P_i$, not vectors.]

1.2. Ontic Theory

A quantum system has a state represented by a vector $|i\rangle$. *If the system passes through an ideal polarizer its state changes to a state* $|o\rangle$ *defining the polarizer. The transition probability for this is*

$$P = \cos^2\theta := |\langle o|i\rangle|^2. \quad (2)$$

1.3. The Miscount

In their mathematical structures, (1) and (2) are identical. There is no way to tell that one theory is sound, and one unsound, from the mathematical theory. We have to watch them in use to discover this.

In the praxic formulation the vectors represent processes. The ontic theory has mistaken them for states, and introduce strange processes like state collapse to compensate for this error.

One sees this when von Neumann speaks of two modes of intervention, **1** and **2**, into a quantum system, representing measurement and propagation (von Neumann, 1932). The count is wrong. There are three buildings, not two, at accelerator facilities, housing three modes of intervention, **I**, **T**, **O**, of inflow, throughflow, and outflow; or beam preparation, target interaction, and counting. These processes are represented by the three factors in the transition amplitude $\langle O|T|I\rangle$. They exist in the simplest quantum experiment too.

The ontic theory did not count the input process because it mistook the process vector $|i\rangle$ for a state. It counts as processes what are merely intervals between processes. A figure-ground reversal has occurred. The ontist imagines an intervention of kind 2 that transforms $|i\rangle$ into $T|i\rangle$, and an intervention of kind **1** that collapses $T|i\rangle$ into $\langle o|$. No such things occur in the laboratory. $|i\rangle$ does not transform into $T|i\rangle$ and then collapse into $|o\rangle$. The process $|i\rangle$ is simply followed by T and then by $\langle o|$, if the quantum reaches the counter.

There is no way to tell from the mathematical theory whether the praxic or ontic theory is right. The error is in the semantics. The use of the theory is described incorrectly.

1.4. Correspondence Principle

Calling a wave function a state violates the correspondence principle. The classical correspondent of a wave function is a Hamilton–Jacobi function. This is not a state but a coordinate transformation.

A photon polarization in flight along an optical bench—say on the z-axis—is postulated by the ontic theory to have a state $\psi(z, t)$ at time t, a unit vector of two complex components, with overall phase ignored.

There is a physical system that has such a state. In the ontic theory, a photon polarization is a particle moving on a 3-sphere with a special first-order dynamical equation; except that unlike such a particle it jumps in a certain probabilistic way when we do what quantum physicists persist in calling a measurement of the polarization along some chosen direction.

The term "measurement" is a misnomer according to the ontic theory, which claims that the process is actually a certain kind of kick of the particle state. We never do what the ontist can call observing the particle, which is to measure its state ψ at some time. What we measure in quantum theory is a Hermitian observable, not a wave function.

1.5. Thought Experiment in Semantics

We should watch the working physicist using the theory to describe the use correctly. The earliest quantum experiment suffices. Malus (1805) considered a photon that has passed undeflected through one crystal of Iceland spar—that is part of the input process $|i\rangle$—and is about to meet another crystal in the outtake process $\langle o|$. Equation (1) is Malus' law for the probability that the photon will again be undeflected.

In the thought experiment we lead a trained quantum physicist to the optical bench and ask her/him to estimate whether a certain photon—say the first after high noon—that has passed undeviated through the first crystal will pass undeviated through the second crystal.

He/she knows not to make any further measurement on said photon in flight between the polarizers, because that could change it and the outcome. He/she might measure the angle θ between the input crystal and the outtake crystal, and use Malus' law. She/he might put a billion other photons through a like process, count the fraction that pass the test, and use that fraction as the probability for the given photon. But in any case her/his practice is not the astronomer's. He can look at the planet itself to tell where it goes and whence it came. She/he must look at the polarizers, not the photon, for the two polarization angles; and nothing she/he can do to the photon will give her/him that information.

We should not call either angle the state of the photon because, whether we call it a state or not, it is not of the photon but of the polarizer.

Transition probability and $|i\rangle$ and $\langle o|$ are features of the experimental process, not of any one product of that process. A ψ does not evolve into a ϕ during a Malus experiment. We choose both freely when we set up the two crystals. They are not the kind of things that evolve but the kind of things that we do.

1.6. Quanta Have No State

It is tempting to take the input process for its product in a first formulation. In classical physics, the input process, the state, and the output process of an allowed transition all determine each other uniquely for purposes of prediction and retrodiction. The classical observer could look at any of them to determine the state.

The quantum physicist, however, does not have that choice, but must note all the processes, which are almost independent of each other in the allowed transitions.

The situation was clearly formulated by Bergmann (1967). The concept of state is inappropriate for quanta.

Quantum theory is a theory of quantum processes. It is no more a theory of a state than special relativity is a theory of the present. This is why Heisenberg called his theory nonobjective and why Blatt and Weisskopf refer to ψ's as channels, not states (Blatt and Weisskopf, 1952). A ψ describes the process, not the product of the process. There is no problem of "collapse" of the state in quantum theory because there is no state to collapse.

The ontic theory loads the photon with an infinity of information, its "state," and denies that the photon can divulge one bit of that information in a measurement. This is the kind of theory that our fathers warned us against. It feigns a hypothesis.

The mathematical problems of the quantum theory all correspond exactly to problems of the ontic theory, but the ontic theory is wrong for the quantum polarization. The natural-language term "state of the system" has a reserved meaning in physics. We err if we call the data that define the state of Venus "the state of Mars" or "the state of the astronomer." We call state of a system something that we can in principle learn from the system itself and that predicts exactly the system's future behavior. A ψ has neither property. Calling a ψ the state creates a discord between quantum practice and ontic principle that creates unease in the ontic physicist, and the better the quantum theory works, the greater the unease. A ψ represents neither a quantum nor the state of one, but a process that produces a quantum.

2. QUANTUM COSMOLOGY

Every quantum field theory since Dirac's quantum electrodynamics has been a quantum theory of the universe. Quantum practitioners generally ignore philosophical problems about self-reference when they make such theories. We argue here that they proceed correctly.

To be sure, these cosmical theories are conspicuously nonoperational. No one in the universe can prepare or register it all sharply. Any physical experimenter is made of the very particles and fields of the theory and so is in the field system, not outside it experimenting on it. For a quantum experimenter, self-measurement would generally be suicidal, unrepeatable, and hence not a measurement.

At first Bohr objected strenuously to a quantum theory of the universe for such reasons. But later he withdrew his objections (Bohr, 1936). It has always been understood, at least tacitly, that one may correspond such theoretical descriptions of the cosmos to those of a physical experimenter, by ignoring the variables that the experimenter ignores, including the vital variables of the experimenter, and noting only those that the experimenter notes.

For those who think that a quantum system has a state, however, it is only one step to thinking that the universe has one, and then one more to wondering what collapses it. This is the ontic fallacy on a cosmic scale.

The quantum cosmology of field theory and the one we use here corresponds—in the sense of Bohr's correspondence principle—to Laplace's classical cosmology and is just as natural. Laplace invented a supreme astronomer who knows the state of the universe. Correspondingly we may imagine a supreme quantum Cosmic Experimenter (CE) who inputs the polarized cosmos with a grand $|I\rangle$ before our experiments and outtakes it with an $\langle O|$ after our experiments. We need the CE to formulate a cosmology as much as Laplace did, but if we want to do quantum cosmology she/he has to be a quantum CE.

These cosmologies are not operational. They deal with fictitious processes carried out by the fictitious CE. We extract operational predictions from them just as Laplace would.

We describe an actual experimenter as a subsystem of the cosmos with its own algebra, and ignore or average over the degrees of freedom of the cosmos that the actual experimenter ignores, especially those of that very experimenter.

If the CE were to have set us up before the fact to do our little experiments and were to read our notebooks after the fact, then her/his readings would be consistent with ours (von Neumann, 1932).

The extracosmic CE is our metaphorical way of allowing for all possible experimenter/system interfaces within one embracing theory. The cosmic algebra is not operational but contains operational algebras as quotients. Quantum theory was already the most relativistic theory we have. Quantum cosmology relativizes it further. It relativizes the experimenter.

Ontists may function in such a quantum cosmology just as they do in laboratory applications. Having imagined a cosmic state-vector, they may imagine a cosmic observer to collapse it by an observation at the end of time.

3. QUNIVAC

Now we describe the structure of the cosmic computer from this extracosmic viewpoint. We begin with the algebra of the universe.

3.1. Quantization Is Stabilization

The standard model of the elementary particles has several non-semisimple groups: groups that are reducible but not decomposable. So does Einstein's model of gravity. All such theories are unstable with respect to small variations in their structure tensors (Inönü and Wigner, 1952; Segal, 1951) called group expansions here. They are singular limits—called group contractions—of deeper, stabler theories that preserve all the basic principles of quantum theory and relativity, at least asymptotically, and are more unified. One of these expanded theories probably fits experiment better than the present contracted theory (Segal, 1951). The contracted theory has probability 0.

One of the deeper instabilities is that of the canonical commutation relations, the differential calculus, and the space–time continuum (Segal, 1951). Group expansion replaces the canonical commutation relations by the Segal relations:

	Newton	Heisenberg	Segal
$\hat{q}\hat{p} - \hat{p}\hat{q} =$	0	hi	$hi,$
$i\hat{q} - \hat{q}i =$	0	0	$h'\hat{p},$
$i\hat{p} - \hat{p}i =$	0	0	$h''\hat{q}$

(3)

Here, since i is anti-Hermitian, we consider antihermitian generators $\hat{q} = iq$ and $\hat{p} = ip$ instead of Hermitian observables $q,\ p$. The Segal ultraquantum commutation relations supplement the Planck quantum constant h with two Segal ultraquantum constants $h',\ h'' \neq 0$. Ultraquantization gives all canonical variables including time and energy simple, discrete uniformly spaced spectra, but still respects the experimental results of the quantum theory as long as $h',\ h''$ are sufficiently small, and preserves orthogonal group symmetries (like Lorentz and de Sitter invariance) exactly.

This discreteness encourages us to postulate an elementary-process hypothesis to replace and unify the continuum hypothesis and the atomic hypothesis: *All physical processes are composed of finitely many finite elementary quantum processes.* (Finkelstein, 1969, 1972, 1996; Wheeler, 1973).

We assert this for space–time translations and boosts as well as particle creation and annihilation. We assume that the elementary quantum process—we call it a chronon for short—lasts at least a minimum time X, and transfers at most a maximum energy $\hbar/X$.

4. REVERSIBLE LOGIC

We assemble qubits into Qunivac by noting how one assembles bits into a classical computer, and assembling quanits into Qunivac similarly.

Since nature is reversible, Qunivac must be reversible in the well known sense of Bennet and Fredkin. To describe a reversible computation we formulate a reversible set theory and logic, classical in this section and quantum in the next.

4.1. Reversible Classical Logic

Classical computers are usually assembled conceptually using class or set algebra. Boole in classical logic and von Neumann in quantum defined a class by an idempotent representing a process of selection $e^2 = e$. Selection, which is filtration, has no inverse. We therefore define a *reversible class* or predicate by a half-wave plate rather than a filter. The operator representing it has eigenvalues 1 and -1. Expressed in terms of the old logical concepts, a reversible predicate is represented by a function assigning $+1$ to elements of the class and -1 to nonelements rather than discarding them.

Set algebra is usually based on operations $\cup$ and $\cap$ which have identity elements but no inverse. These are unsuitable for a reversible logic or a reversible universe. For a reversible set algebra we replace them by reversible operations, which therefore form a group (assuming associativity here).

The only two reversible logical operations on truth values are XOR and its negation EQUALS. We arbitrarily choose XOR as the more familiar of the two. The XOR of two reversible predicates is their arithmetic product as functions from elements to ± 1. Then 1 must represent the empty set, which is the identity for XOR.

We still need an element of structure to define the complement, to tell a unit class from its complement, and to define the universal class.

We introduce a $\mathbf{Z}$-valued logical grade $|x|$ for this purpose. This is just the modulus of the old class logic: The grade is 0 for the null set, 1 for unit sets, and so on. For disjoint sets only, $|xy| = |x| + |y|$.

Our reversible classical logic (or set theory) is thus a graded group of unipotent elements $x^2 = 1$.

4.2. Reversible Quantum Logic

To go from the classical to the quantum logic we turn to superposition. The reversible quantum class or set algebra is then the linear associative real algebra generated by the unit sets, which are still unipotent: $x^2 = 1$. In a relativistic theory we generalize to as x^2 of ± 1. This is the Clifford algebra generated by the unit classes or sets (Finkelstein, 1982). A reversible quantum logic is a Clifford logic.

There are a number of indicators that the Clifford logic should replace the von Neumann one in fundamental studies. Stability in the sense of Irving Segal; simplicity in the algebraic sense, which is closely related; spinoriality, which goes with every Clifford algebra; and now reversibility, which we had not noticed before.

Class algebra, reversible or not, is the boring part of set theory for software architects. Computation usually has a hierarchic structure, for forming programs composed of subprograms. The creative process is setting up the hierarchic structure. Since nature has a hierarchic structure Qunivac must have one too.

The hierarchic operation of classical set theory is the power set functor $\mathcal{P} : X \mapsto 2^X$. Set calculus is the theory of the iterated power set functor. We use the power set to organize the computer, to construct its organs.

Our quantum power set functor is the Clifford algebra functor Cliff: $A \mapsto 2^A$ from graded algebras to graded algebras.

Till recently (Finkelstein, 1996), I still used Grass, not Cliff, to form a set calculus, but that theory is unstable and irreversible, and the Clifford logic is its stabilization (Baugh *et al.*, 2001).

We call a quantum aggregate described by a Clifford algebra over the one-quantum algebra a *squad*. The qubits of a squad obey a real variant of Wilczek's Clifford statistics (Nayak and Wilczek, 1996; Wilczek, 1998; Wiczek and Zee, 1982).

If the qubit of Qunivac has a (real) algebra A, the algebra of observables of a squad of qubits, say the entire Qunivac, is the Clifford algebra $\mathbf{2}^A$. The grade of this algebra counts elementary computer operations, which have grade 1.

The elements of the classical set calculus are built inductively from 1 (the empty set) with Peano's unitizer or successor operator $\iota z = \{z\}$, $\iota : X \to 2^X$. Our quantum $\iota : C \to \mathbf{2}^C$ is a linear morphism $C \to \mathbf{2}^C$ that transforms any element $z \in C$ into a first-grade element $\iota z \in C$ and reverses the norm $\|z\| := \mathrm{Re}\, z^2$:

$$(\iota z)^2 = -\|z\|.\|\iota z\| = -\|z\|. \tag{4}$$

We introduce this sign reversal to generate the indefinite metrics needed for relativity and gauge theory.

The operator ι has no inverse since most sets are not unit sets, but is reversible in that any unit set has a unique element, so that ι has a left inverse $\iota^{\llcorner} : \iota^{\llcorner}\iota = \text{ld}$. We fix $\iota^{\llcorner}$ uniquely by setting it to 0 on all sets of sharp grades other than 1.

Then we define the reversible quantum logic $C = \text{Cliff}(\iota)$, the real Clifford algebra generated by ι. Since one calls a Clifford algebra with four units the quaternions, we might call Cliff(ι) the infinions.

This provides new content to the old surmise (Finkelstein, 1969, 1972) that the quantum universe is a quantum computer.

We suppose that in any possible universe a cosmical but finite number N of anticommuting binary variables suffices, generating a finite-dimensional subalgebra of the infinions that can be called the cosmonions.

The Clifford sum provides Qunivac with the famous quantum parallelism that lets it compute so fast.

Iterated Clifford-algebra formation provides a hierarchy-generating, or subprogram-forming, function for Qunivac.

A mode of Qunivac is then represented by a spinor of the cosmonion algebra.

4.3. Fermions

We have programmed Qunivac for a Dirac particle in a quantum space–time (Galiautdinov and Finkelstein, 2001). It respects Lorentz invariance exactly. Its quantification preserves and strengthens the observed spin–statistics correlation, now giving it a purely algebraic origin.

On the other hand Qunivac beats the standard Heisenberg uncertainty relations. Position and momentum are now proportional to angular momentum operators in higher dimensions and so is their commutator η (Galiautdinov and Finkelstein, 2001). All three can be exactly 0 at the same time in a singlet channel. We expect that as in quaternion quantum field theory, η contracts to the Higgs field in the limit $X \to 0$, $N \to \infty$ and $i\hbar$ is its effective value in the vacuum. The usual Heisenberg indeterminacy relations appear to be good approximations only for values of η (hopefully, the Higgs field) close to its vacuum value.

At high energy, $\sim\hbar/X$, Qunivac also violates the usual continuum-based locality principle. Elementary processes connect events separated not infinitesimally as Einstein postulated but by a time χ (Chi), the chronon time. At energies much lower than $\hbar/X$ this would not show up strongly in the experimental data.

The simplest stabilization of the Dirac equation predicts an upper bound $\hbar/(Xc^2)$ on the mass of elementary fermions. If we tentatively identify this limit with M(Top quark), we can estimate X. The distance cX is then two or three orders of magnitude smaller than Dehmelt's estimate of the electron size (Dehmelt, 1998).

It is conceivable that Dehmelt's form factor size and our X are both right. This would, however, imply that the electron is quite composite, as Dehmelt proposes.

X is many orders of magnitude greater than the Planck time. This need not trouble us. The Planck time comes from two unstable contracted theories that may lack X. We propose that X is a more serious limit arising from the ultraquantum structure of space–time and that the gravitational field is a condensate like the Higgs field.

4.4. Fields

Field theory begins with a partition of variables into field and space–time. The space–time variables are of the experimenter, the field variables are of the system. The set of fields is locally an exponential Y^X, where Y is the field fiber and X is the space–time.

Programming Qunivac for field theory requires us to define the set exponential Y^X when the field variable space Y and the space–time X are both quantum, with Clifford algebras for their operator algebras. We insist on the correspondence principle. Our construction must have a reasonable classical limit.

To represent a field Clifford algebraically we first represent the universe as a squad of events $U = \mathbf{2}^E$. Then we reduce each event E to a pair $E = y \otimes X$. X gives its location and y describes a "filament" at X, so called because the fiber of this quantum field bundle will be a squad of y's. This factorization irepresents a condensation that reduces the orthogonal group of the event E to a product of two smaller orthogonal groups. We then define the field algebra $U = Y^X := \mathbf{2}^{y \otimes X}$. The field fiber at each point is clearly $Y = \mathbf{2}^y$: a squad of filaments indeed.

This construction is possible and easy when and apparently only when the field algebra Y at each point has the form $Y = \mathbf{2}^y$. This happens to work for spinor fields.

Since we have formulated this quantum field entirely in Clifford algebra, it is easy to see a classical limit. One simply replaces **2** by 2 throughout and recovers discrete field theory.

In an earlier quaternionic theory the varying $i\hbar$ provided the Higgs field and reduced the gauge group. Now the quaternions have spawned a cosmological number of Clifford units and the question is reopened.

It seems possible that the cosmos can be described as an ultraquantum logical engine. In von Neumann's classic treatise on quantum theory (von Neumann, 1932), he interpreted quantum theory as a revised physical logic, which shaped my subsequent research. When he was asked to update Hilbert's famous list of problems for the Fifth International Congress of Mathematicians in 1954 he issued but one challenge, to further explore that quantum-physical logic. Infinite-dimensional problems in first-order logic aroused von Neumann's mathematical interest, and here we face finite questions in higher-order logic. I hope these elementary developments stimulate other physicists too to explore von Neumann's rich legacy.

ACKNOWLEDGMENTS

I thank James Baugh, André Galiautdinov, and Mohsen Shiri for extended discussions of these ideas, Frank "Tony" Smith for algebraic information, Frank Wilczek for his studies of Clifford statistics, and Shlomit Ritz Finkelstein for discussions and much help with the presentation.

REFERENCES

Bacon, F. (1994). In *Novum Organum*, Articles XXXVIII, XXXIX. P. Urbach and J. Gibson, trans. and ed., Open Court Publishing, Peru. (Original work published 1620)

Baugh, J., Finkelstein, D., Galiautdinov, A., and Saller, H. (2001). Clifford algebra as quantum language. *Journal of Mathematical Physics*. **42**, 1489–1500.

Bergmann, P. G. (1967). The quantum state vector and physical reality. In *Studies in the Foundations, Methodology, and Philosophy of Sciences. Vol. 2: Quantum Theory and Reality.* M. Bunge, ed., Springer, Heidelberg, Germany, Ch. 3.

Blatt, J. M. and Weisskopf, V. F. (1952). *Theoretical Nuclear Physics*. Wiley, New York.

Bohr, N. (1936). Causality and complementarity. *Philosophy of Science* **4**, 293–294. Address before the Second International Congress for the Unity of Science, June, 1936.

Dehmelt, H. G. (1998). Nobel address. *Nobel Lectures in Physics*. World Scientific, 1998.

Finkelstein, D. (1969). Space-time code, *Physical Review* **184**, 1261–1271. Finkelstein, D. (1972). Space-time code II, *Physical Review D* **184**, 4–328.

Finkelstein, D. (1982). Quantum set theory and Clifford algebra. *International Journal of Theoretical Physics* **21**, 489–503.

Finkelstein, D. (1996). *Quantum Relativity*. Springer, Heidelberg, Germany.

Finkelstein, D. R. and Galiautdinov, A. A. (2001). Cliffordons. *Journal of Mathemetical Physics* **42**, 3299–3314.

Galiautdinov A. A. and Finkelstein, D. R. (2001). Non-local corrections to the Dirac equation. hep-th/0106273 [LANL].

Inönü, E. and Wigner, E. P. (1952). On the contraction of groups and their representations. *Proceedings of the National Academy of Sciences* **39**, 510–525.

Malus, E. L. (1805). *Mémoires de la Société d'Arcueil* **2**, 143. Excerpted in Magie, W. F. (1935). *A Source Book in Physics*, McGraw–Hill, New York.

Nayak, C. and Wilczek, F. (1996). *Nuclear Physics B* **479**, 529.

von Neumann, J. (1932). *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin. *Mathematical Foundations of Quantum Mechanics*. Translator R. T. Beyer. Princeton, 1955.

Segal, I. E. (1951). A class of operator algebras which are determined by groups. *Duke Mathematics Journal* **18**, 221.

Wheeler, J. A. (1973). The elementary quantum act as higgledy-piggledy building mechanism. In *Quantum Theory and the Structure of Time and Space. IV.* L. Castell, M. Drieschner, and C. F. V. Weizsaĉker, eds., Hanser, Munich, pp. 27–30.

Wilczek, F. (1998). Projective statistics and spinors in Hilbert space, hep-th/9806228 [LANL].

Wilczek, F. and Zee, A. (1982). Families from spinors. *Physical Review D* **25**, 553.